The Linear Sigma-Model in the $1/N$-Expansion via Dynamical Boson Mappings and Applications to $\pi\pi$-Scattering

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Abstract

We present a non-perturbative method for the study of the $O(N+1)$-version of the linear sigma-model. Using boson-mapping techniques, in close analogy to those well-known for fermionic systems, we obtain a systematic $1/N$-expansion for the Hamiltonian which is symmetry-conserving order by order. The leading order for the Hamiltonian is evaluated explicitly and we apply the method to $\pi\pi$-scattering, in deriving the T-matrix to leading order.

1 Introduction

In a recent publication [1] we have presented a non-perturbative treatment of the pion as a Goldstone particle within the linear sigma-model using the Hartree-Fock-Bogoliubov (HFB)-RPA theory. Our initial interest in a non-perturbative approach was motivated by the desire to find a pion-pion scattering equation which obeys unitarity while being consistent with constraints from chiral symmetry, such as the vanishing of the a_0^0 scattering length in the chiral limit, $m_\pi \to 0$. This is nontrivial since, even in cases where the tree-level scattering amplitude fulfills the chiral symmetry requirements, the iteration through various reduction schemes of the Bethe-Salpeter equation, such as time-ordered perturbation theory or the Blankenbecler-Sugar reduction breaks the consistency [2]. Unitary scattering equations are a necessity, however, when studying medium modifications of elementary cross sections. Indeed for example a π^+-π^- pair in a hot pion gas may form a resonance close to the $2m_\pi$ threshold or even a bound state as was discussed in [3, 5, 4]. Of course such features cannot be treated in chiral perturbation theory. In the present work we will therefore generalize the theory to the two-pion problem with application to π-π scattering and will indeed find a scattering equation which fulfills the requirements of chiral symmetry.
In constructing the physical pion we have used in our previous work [1] a RPA operator which mixes a quasi-pion with a quasi-pion quasi-sigma pair, obtained from a HFB mean-field calculation, in such a way that this operator contains the axial charges Q_5^i as a special limit. The resulting RPA-eigenmode, at zero three-momentum, is then identified with the physical pion and we were able to show that it becomes a Goldstone mode in the chiral limit. To fully restore chiral symmetry, which is broken at the HFB level, it is important to write down the RPA operator in the selfconsistent HFB basis whose vacuum is characterized by the coherent (squeezed) state $|\Phi\rangle \propto \exp\left[(\sum_q z_1(q)\vec{a}_q^+ \cdot \vec{a}_{-q}^+ + z_2(q)b_q^+ b_{-q}^+) + wb_0^+\right]|0\rangle$. The z's and w are variational parameters and a^+, b^+ are the original pion and sigma creation operators corresponding to the vacuum defined by $a_q^i|0\rangle = b_q|0\rangle = 0$. Although this approach is fairly standard and described in most textbooks (see e.g. [6, 7]) the field theoretical context exhibits some particularities which are worthwhile mentioning. The main point is that the selfconsistent HFB basis generates a mass for the quasipion even in the chiral limit. Despite the fact that they emerge from a respectable Raleigh-Ritz variational principle and that Goldstone's theorem is, as it should be, perfectly restored once the RPA fluctuations are included this remains a somewhat disturbing feature. In particular for the two-pion problem a serious problem emerges since the asymptotic states cannot be defined easily in a way which is consistent which chiral symmetry. The HFB quasiparticles can clearly not be used. Even though the two-pion RPA, when solved in a similar fashion as the single-pion RPA, will exhibit a spurious mode in the chiral limit, this does not imply that the scattering length will vanish. Interactions will persist so as to cancel the quasiparticle mass term, in close analogy to the single-pion case. This is obviously unphysical. In addition, with explicit symmetry breaking, the threshold will be at twice the quasipion mass and not at twice the physical mass. Because of the obvious advantages of a Raleigh-Ritz variational theory it maybe worthwhile for the future to consider modifications of the trial wave function

to guarantee that Goldstone bosons remain as such. In this paper we will not pursue this option despite the fact that the approach of [1] might probably be generalized to the two-pion problem. Instead we will identify the origin of the mass generation in the HFB mean field by studying the $O(N+1)$ version of the linear sigma-model. This will lead to a systematic $1/N$-expansion technique which has the advantage of being non-perturbative as well (it will also lead to RPA type equations). It will be shown that, to leading order, the pion is a Goldstone boson opening the possibility to also derive a $\pi - \pi$ scattering equation to leading order which is physically meaningful. The disadvantage is that the Raleigh-Ritz variational principle is lost. In regards to earlier work on non-perturbative approaches to the Φ^4 field theories in general [8, 9] and the $1/N$-expansion in particular [10, 11, 12] we here will however present a novel approach. It employs Boson expansion methods and relies on work in nuclear physics where such techniques, in connection with the restoration of spontaneously broken symmetries, have been studied extensively [13, 14]. Boson expansion methods for boson pairs in interacting Bose systems have first been introduced by Curutchet, Dukelsky, Dussel, and Fendrik [15] and further elaborated by Bijker, Pittel and Dukelsky in [16], in close analogy to the techniques known from the fermion case [6, 7].
Since the subject of restoration of broken symmetries is full of subtleties, it seems worthwhile to present this approach in some detail. Our aim will be to derive a covariant scattering equation for two pions to-leading order without taking the large-N limit. This will be achieved by a systematic $1/N$-expansion of the Hamiltonian of the linear sigma-model. Of course in reality $N = 3$ and it will be necessary to work out corrections to the lowest order $\pi - \pi$ scattering equation for quantitative studies. Within the bosonization framework this can be done systematically and we will leave the evaluation of the $1/N$ corrections for future work.
In detail our paper is organized as follows. First we recapitulate the most important results of our previous work, here applied, however, to the $O(N+1)$-version of the linear sigma-model. As mentioned above, this mainly serves to elucidate the origin of the mass generation at the HFB level. In sect. 3 the boson mapping techniques are introduced which lead to the $1/N$ expansion of the sigma-model Hamiltonian. The leading-order results are then represented. For the two-pion sector the T-matrix will be extracted in sections 4 and 5 where we also comment on the symmetry properties. Conclusions and an outlook will be given in sect. 6.

2 HFB-RPA for the $O(N+1)$ Sigma-Model

As mentioned in the introduction this section serves to identify the origin of the mass generation in the HFB-RPA approach of ref. [1] from the point of view of a $1/N$-expansion. We therefore consider the $O(N+1)$ version of the linear sigma-model. The difference to the $SU(2) \times SU(2) \sim O(4)$ case lies in the fact that one now has an N-component isovector pion field. The Lagrangian density then reads

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \vec{\pi})^2 + (\partial_\mu \hat{\sigma})^2\right] - \frac{\mu_0^2}{2}\left[\vec{\pi}^2 + \hat{\sigma}^2\right] - \frac{\lambda_0^2}{4N}\left[\vec{\pi}^2 + \hat{\sigma}^2\right]^2 + \sqrt{N} c \hat{\sigma} \tag{1}$$

where λ_0 represents the bare coupling constant, μ_0 the bare mass parameter and π and $\hat{\sigma}$ denote the bare pion and sigma fields, respectively. Chiral symmetry is explicitly broken (in the PCAC sense) by the last term in the Lagrangian, $c\hat{\sigma}$.

It is now convenient to define the field operators in terms of creation and annihilation operators as

$$\begin{aligned} \vec{\pi}(\mathbf{x}) &= \int \frac{d^3q}{\sqrt{(2\pi)^3 2\omega_q}} \left(\vec{a}_q e^{i\mathbf{qx}} + \vec{a}_q^+ e^{-i\mathbf{qx}} \right) \\ \hat{\sigma}(\mathbf{x}) &= \int \frac{d^3q}{\sqrt{(2\pi)^3 2\omega_q}} \left(b_q e^{i\mathbf{qx}} + b_q^+ e^{-i\mathbf{qx}} \right) \end{aligned} \tag{2}$$

where the frequency ω_q, common to both fields, is given by

$$\omega_q = \sqrt{\mu_0^2 + q^2}. \tag{3}$$

Next a canonical transformation is performed for the pion- as well as the sigma- fields by introducing a new set of creation and annihilation operators through a Bogoliubov rotation

$$\begin{aligned} \vec{\alpha}_q^+ &= u_q \vec{a}_q^+ - v_q \vec{a}_{-q}, \\ \beta_q^+ &= x_q b_q^+ - y_q b_{-q} - w^* \, \delta_{q0} \end{aligned} \tag{4}$$

with u_q, v_q, x_q and y_q being even functions of their argument, and w a c-number. The additional 'shift' in the second equation accounts for the macroscopic condensate

$$\langle \hat{\sigma} \rangle = \frac{\langle b_0^+ \rangle + \langle b_0 \rangle}{\sqrt{(2\pi)^3 2\mu_0}} = \frac{(x_0 + y_0)(w + w^*)}{\sqrt{(2\pi)^3 2\mu_0}} = \sqrt{N} s \tag{5}$$

To render the transformations canonical the Bogoliubov factors have to obey the constraints

$$u_q^2 - v_q^2 = 1, \qquad x_q^2 - y_q^2 = 1. \tag{6}$$

The quasiparticle vacuum $|\Phi\rangle$ ($\vec{\alpha}|\Phi\rangle = \beta|\Phi\rangle = 0$) is now given by the following coherent state

$$|\Phi\rangle = \exp \left[\sum_q (z_1(q) \vec{a}_q^+ \cdot \vec{a}_{-q}^+ + z_2(q) b_q^+ b_{-q}^+) + \frac{w}{x_0} b_0^+ \right] |0\rangle. \tag{7}$$

were $|0\rangle$ denotes the vacuum for the original basis ($\vec{a}_q|0\rangle = b_q|0\rangle = 0$) and $z_1 = \frac{v}{2u}$, $\quad z_2 = \frac{y}{2x}$.
It is straightforward to write the Hamiltonian in the quasiparticle basis. The amplitudes u, v, x, y as well

as the condensate s are determined by minimizing the vacuum expectation value $\langle\Phi|H|\Phi\rangle/\langle\Phi \mid \Phi\rangle$. This is equivalent to demanding that the single-particle part of H is diagonal and leads to the well-known BCS gap equations and an equation which determines s. These can be cast in a form which contains physical quantities:

$$\begin{aligned} \mathcal{E}_\pi^2 &= \mu_0^2 + \lambda_0^2 \left[\frac{N+2}{N} I_\pi + \frac{1}{N} I_\sigma + s^2\right] \\ \mathcal{E}_\sigma^2 &= \mu_0^2 + \lambda_0^2 \left[I_\pi + \frac{3}{N} I_\sigma + 3s^2\right] \\ \frac{c}{s} &= \mu_0^2 + \lambda_0^2 \left[I_\pi + \frac{3}{N} J_\sigma + s^2\right]. \end{aligned} \tag{8}$$

where $\mathcal{E}_\pi$ and $\mathcal{E}_\sigma$ denote the masses of the quasi-pion and quasi-sigma respectively and I_π and I_σ are loop integrals

$$I_\pi = \int \frac{d^4q}{(2\pi)^4} \quad \frac{i}{q^2 - \mathcal{E}_\pi^2 + i\eta}, \qquad I_\sigma = \int \frac{d^4q}{(2\pi)^4} \quad \frac{i}{q^2 - \mathcal{E}_\sigma^2 + i\eta}. \tag{9}$$

For $N = 3$ these results coincide with those of ref. [1]. By rewriting the equations for the quasiparticle masses as

$$\begin{aligned} \mathcal{E}_\pi^2(0) &= \frac{c}{s} + \frac{2\lambda_0^2}{N}\left[I_\pi - I_\sigma\right], \\ \mathcal{E}_\sigma^2(0) &= \frac{c}{s} + 2\lambda_0^2 s^2. \end{aligned} \tag{10}$$

one sees that, in the chiral limit ($c \to 0$), the quasipion mass remains massive due to the nonvanishing difference $I_\pi - I_\sigma$. At first glance this seems to be a $1/N$ effect, however due to the selfconsistency one actually has a power series in $1/N$!

To render the pion massless one needs to go further and include a set of well-defined RPA fluctuations. The procedure is described in detail in ref. [1] and thus we will be very brief here. One introduces an excitation operator with the quantum numbers of the pion which is, at most, bilinear in the creation and destruction operators

$$\vec{Q}_\pi^+ = X_\pi^1 \vec{\alpha}_0^+ \; - \; Y_\pi^1 \vec{\alpha}_0 \quad + \quad \sum_q \left[X_\pi^2(q)\beta_q^+ \vec{\alpha}_{-q}^+ \; - \; Y_\pi^2(q)\beta_{-q}\vec{\alpha}_q\right] \tag{11}$$

and similarly for the sigma field

$$Q_\sigma^+ = \left[X_\sigma^1 \beta_0^+ \; - \; Y_\sigma^1 \beta_0\right] + \sum_q \left[X_\sigma^2(q)\beta_q^+\beta_{-q}^+ \; - \; Y_\sigma^2(q)\beta_{-q}\beta_q\right] + \sum_q \left[X_\sigma^3(q)\vec{\alpha}_q^+ \cdot \vec{\alpha}_{-q}^+ \; - \; Y_\sigma^3(q)\vec{\alpha}_{-q} \cdot \vec{\alpha}_q\right]. \tag{12}$$

The RPA ground state is determined by $\vec{Q}_\pi|RPA\rangle = Q_\sigma|RPA\rangle = 0$. Within the equation of motion method [7, 17] the RPA frequencies can now be determined which finally leads to the following equation for the physical pion mass

$$m_\pi^2 = \frac{c}{s} + \frac{2\lambda_0^2}{N} \frac{\left[\mathcal{E}_\pi^2 - \mathcal{E}_\sigma^2\right]\left[\Sigma_{\pi\sigma}(0) - \Sigma_{\pi\sigma}(m_\pi^2)\right]}{1 - \frac{2\lambda_0^2}{N}\Sigma_{\pi\sigma}(m_\pi^2)} \tag{13}$$

where $\Sigma_{\pi\sigma}(p^2)$ is the quasipion-quasisigma bubble given by

$$\Sigma_{\pi\sigma}(p^2) = -i\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - \mathcal{E}_\pi^2 + i\eta}\,\frac{1}{(p-q)^2 - \mathcal{E}_\sigma^2 + i\eta}. \tag{14}$$

In the chiral limit ($c = 0$) the Goldstone theorem is now manifest since eq. (13) has a zero-energy solution. Here again one sees that the RPA has brought about a power series in $1/N$ which, in the chiral limit, cancels exactly the mean field contributions in such a way that the symmetry is restored.
After the inclusion of RPA fluctuations one obtains for the sigma mass

$$m_\sigma^2 = \mathcal{E}_\sigma^2 + 2\lambda_0^4 s^2 \frac{\Sigma_{\pi\pi}(m_\sigma^2) + \frac{9}{N}\Sigma_{\sigma\sigma}(m_\sigma^2) - 6\lambda_0^2\frac{N+3}{N^2}\Sigma_{\pi\pi}(m_\sigma^2)\Sigma_{\sigma\sigma}(m_\sigma^2)}{\left[1 - \frac{N+2}{N}\lambda_0^2\Sigma_{\pi\pi}(m_\sigma^2)\right]\left[1 - \frac{3}{N}\lambda_0^2\Sigma_{\sigma\sigma}(m_\sigma^2)\right] - \frac{1}{N}\lambda_0^4\Sigma_{\pi\pi}(m_\sigma^2)\Sigma_{\sigma\sigma}(m_\sigma^2)} \tag{15}$$

with the $\pi\pi$ and $\sigma\sigma$ bubbles given by

$$\begin{aligned} i\Sigma_{\pi\pi}(p^2) &= \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - \mathcal{E}_\pi^2 + i\eta}\,\frac{1}{(p-q)^2 - \mathcal{E}_\pi^2 + i\eta} \\ i\Sigma_{\sigma\sigma}(p^2) &= \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - \mathcal{E}_\sigma^2 + i\eta}\,\frac{1}{(p-q)^2 - \mathcal{E}_\sigma^2 + i\eta} \end{aligned} \tag{16}$$

It should be mentioned that the condensate s does not receive any contributions from RPA fluctuations in this approximation.

In spite of the fact that the Goldstone theorem is fulfilled by the formalism described above it has a severe draw back for the $\pi - \pi$ scattering problem. As mentioned in the introduction there will be interactions in the chiral limit due to the fact that the quasipions remain massive.
It is now interesting to consider the limit ($N \to \infty$). By inspection of eqs. (8),(13)and (15) the HFB-RPA approach yields the following results

$$\begin{aligned} \frac{c}{s} &= \mu_0^2 + \lambda_0^2\left[I_\pi + s^2\right] \\ m_\pi^2 &= \mathcal{E}_\pi^2 = \frac{c}{s} \end{aligned}$$

$$\mathcal{E}_\sigma^2 = \frac{c}{s} + 2\lambda_0^2 s^2$$
$$m_\sigma^2 = \mathcal{E}_\sigma^2 + 2\lambda_0^4 s^2 \frac{\Sigma_{\pi\pi}(m_\sigma^2)}{[1 - \lambda_0^2 \Sigma_{\pi\pi}(m_\sigma^2)]} \tag{17}$$

where the $\Sigma_{\pi\pi}$ bubble is built out of the asymptotic states

$$\Sigma_{\pi\pi}(p^2) = -i \int \frac{d^4q}{(2\pi)^4} D_\pi(q) D_\pi(p-q) \tag{18}$$

which are 'acceptable' from the point of view of chiral symmetry. Obviously the pion is now a Goldstone boson already at the mean-field level. The RPA fluctuations do not survive in the large-N limit in the case of the pion while for the sigma they are still partly present. In fact, as will become clear in sect. 3, the result obtained for the pion mass corresponds exactly to the Hartree-Bogoliubov solution since terms induced by the Bose statistics disappear. The above results have been obtained previously by several authors using various methods [10, 11]. What we have presented so far is yet another way of deriving these results. They merely serve as a motivation for introducing a systematic $1/N$ expansion in the next section.

First we wish to address another issue, however. In contrast to the HBF-RPA approach in $O(4)$ the large-N limit permits a relatively straightforward regularization. The bare coupling constant λ_0 and bare mass μ_0 are replaced by renormalized quantities:

$$\lambda^2 = \frac{\lambda_0^2}{1 - \lambda_0^2 L_0(\Lambda)} \qquad \mu^2 = \mu_0^2(1 + \lambda^2 L_0(\Lambda)) + \lambda^2 K_0(\Lambda) \tag{19}$$

where L_0 and K_0 represent a logarithmic divergence and a quadratic divergence, respectively, and Λ is a four-dimensional cut-off. Defining the regularized pionic tadpole and the $\pi\pi$ bubble as

$$\bar{I}_\pi = I_\pi - K_0(\Lambda) - m_\pi^2 L_0(\Lambda) \qquad \bar{\Sigma}_{\pi\pi}(p^2) = \Sigma_{\pi\pi}(p^2) - L_0(\Lambda) \tag{20}$$

the physical pion- and sigma masses are rendered finite

$$m_\pi^2 = \frac{c}{s} = \mu^2 + \lambda^2 \left[\bar{I}_\pi + s^2\right]$$
$$m_\sigma^2 = \frac{c}{s} + \frac{2\lambda^2 s^2}{[1 - \lambda^2 \bar{\Sigma}_{\pi\pi}(m_\sigma^2)]}. \tag{21}$$

In addition, the well-known Ward identity for the $\sigma\pi\pi$ vertex [18] is obeyed:

$$m_\sigma^2 - m_\pi^2 = \frac{2\lambda^2 s}{[1 - \lambda^2 \bar{\Sigma}_{\pi\pi}(m_\sigma^2)]}\, s. \tag{22}$$

A comment is in order here. From eq. (19) one can easily see that the renormalized coupling constant λ vanishes when the cut-off Λ is removed and hence the theory becomes trivial. The triviality of the $\lambda\Phi^4$-theory in the $1/N$ approach has been thoroughly studied (see for instance [11, 12]) and, to leading order, it has been shown in [11] that the vacuum has the same group of invariance as the Lagrangian. The same is evidently true here. This problem may be avoided, however, by assigning a physical significance to a finite cut-off Λ, arguing that the sigma-model only describes physics up to a certain scale. In this spirit it was shown in ref. [19] that, with a cutoff version of the $\lambda\Phi^4$-theory, the nontrivial vacuum can be stabilized. Another interesting possibility is the introduction of more degrees of freedom (vector bosons in the case of $\lambda\Phi^4$ [20]) which may have its own difficulties, however. Clearly more work will be needed on this point.

3 Dynamical Boson Mapping and the $1/N$-Expansion

In this section we present an approach to the $1/N$-expansion in the cut-off version of the $O(N+1)$ sigma-model inspired by well-known boson mapping techniques used in nuclear physics [6, 7, 14]. This method is systematic in the sense that it represents a well-defined expansion in powers of $1/\sqrt{N}$ and no a posteriori limit $N \to \infty$ has to be taken. In fact, the RPA is based on the so-called quasi-boson approximation which, in the nuclear physics context, means that Fermion pairs are replaced by ideal bosons. Analogously the RPA for bosons, as applied in the last section, implies the bosonization of boson pairs. Boson expansion techniques for interacting bose systems have first been studied by Curutchet, Dukelsky, Dussel, and Fendrik (CDDF) in [15] and by Bijker, Pittel and Dukelsky (BPD) in [16]. We will apply this technique to obtain a systematic $1/N$ expansion of the Hamiltonian of the linear sigma-model. It has been discussed at length by Marshalek et al. [13, 14] how such an expansion preserves at the same time the symmetries order by order. We are specifically interested in the $\pi-\pi$ scattering problem for which we want to establish a Lippmann-Schwinger type of equation which fulfills unitarity as well as constraints from chiral symmetry such as the vanishing of the $\pi-\pi$ scattering length once the explicit symmetry breaking is removed ($m_\pi \to 0$). To proceed one introduces ideal bose operators $(A^+_{q,p}, A_{q,p})$ and bosonizes pion pair operators as first laid out by Holstein and Primakoff [21] (in nuclear physics it has become known also as the Belyaev-Zelevinsky method [22]). The method consists of the following mapping

$$\begin{aligned}
\vec{a}_q^+\vec{a}_p^+ &= \left(A^+\sqrt{N + A^+A}\right)_{q,p} \\
\vec{a}_q\vec{a}_p &= \left(\vec{a}_p^+\vec{a}_q^+\right)^+ \\
\vec{a}_q^+\vec{a}_p &= \left(A^+A\right)_{q,p}
\end{aligned} \tag{23}$$

where the new operators $(A^+_{q,p}, A_{q,p})$ are ideal bosons and obey the usual Heisenberg-Weyl algebra:

$$\begin{aligned}\left[A_{m,n}\,,\,A^+_{p,q}\right] &= (\delta_{m,q}\delta_{n,p}+\delta_{n,q}\delta_{m,p})\\ \left[A^+_{m,n}\,,\,A^+_{p,q}\right] &= [A_{m,n}\,,\,A_{p,q}] \,=\, 0.\end{aligned} \tag{24}$$

With respect to the more familiar Fermion case one should notice the change of sign on the *rhs* of the first equation in eq. (24) and under the square root in eq. (23). In addition, the operators $A_{q,p}$ are symmetric under exchange of indices

$$A_{q,p} \,=\, A_{p,q} \qquad\qquad A^+_{q,p} \,=\, A^+_{p,q} \tag{25}$$

rather than antisymmetric as in the case of Fermion bosonization. The approach of Belyaev-Zelevinski to the bosonic mapping requires the realization of the original algebra for pairs of pions

$$\begin{aligned}\left[\vec{a}_1\vec{a}_2\,,\,\vec{a}^+_3\vec{a}^+_4\right] &= N\left(\delta_{13}\delta_{24}+\delta_{14}\delta_{23}\right)\,+\,\delta_{13}\vec{a}^+_4\vec{a}_2+\delta_{23}\vec{a}^+_4\vec{a}_1+\delta_{14}\vec{a}^+_3\vec{a}_2+\delta_{24}\vec{a}^+_3\vec{a}_1\\ \left[\vec{a}_1\vec{a}_2\,,\,\vec{a}^+_3\vec{a}_4\right] &= \delta_{13}\vec{a}_2\vec{a}_4+\delta_{23}\vec{a}_1\vec{a}_4\\ \left[\vec{a}^+_1\vec{a}^+_2\,,\,\vec{a}^+_3\vec{a}^+_4\right] &= \left[\vec{a}_1\vec{a}_2\,,\,\vec{a}_3\vec{a}_4\right]\,=\,0.\end{aligned} \tag{26}$$

The reader may verify that this is in fact true order by order, using the Holstein-Primakoff mapping in eq. (23).

To write the Hamiltonian in the (CDDF-BPD)-representation we need to define the sigma field. Since the latter is to be constructed perturbatively, it is legitimate to use a form similar to eq. (2)

$$\hat{\sigma}(\mathbf{x}) \,=\, \int \frac{d^3q}{\sqrt{(2\pi)^3 2\mathcal{E}_\sigma(q)}}\left(b_{\mathbf{q}}e^{i\mathbf{qx}}+b^+_{\mathbf{q}}e^{-i\mathbf{qx}}\right) \tag{27}$$

with the sigma-quasiparticle mass $\mathcal{E}_\sigma$ to be defined later. With these definitions the Hamiltonian takes the following form:

$$\begin{aligned}H \;=\;& \mathcal{H}_0 \,+\, \int\frac{d^3q}{2\mathcal{E}_\sigma(q)}\left[\left(\mathcal{E}^2_\sigma(q)+\omega^2_q+\lambda^2_0 I_0\right)b^+_q b_q-\frac{1}{2}\left(\mathcal{E}^2_\sigma(q)-\omega^2_q-\lambda^2_0 I_0\right)\left(b^+_q b^+_{-q}+b_{-q}b_q\right)\right]\\ &+\, \int d^3q \quad e_0(q)\left(A^+A\right)_{q,q}+\frac{\Delta_0(q)}{2}\left[\left(A^+\sqrt{N\,+\,A^+A}\right)_{q,-q}+\left(\sqrt{N\,+\,A^+A}A\right)_{-q,q}\right]\\ &+\, \int d\mathbf{x}\quad\left[\frac{\lambda^2_0}{4N}\left(:\,\vec{\pi}^{\mathbf{2}}(\mathbf{x})\,:+\hat{\sigma}^2(x)\right)^2\,-\,\sqrt{N}c\hat{\sigma}(x)\right]\end{aligned} \tag{28}$$

where the colon indicates normal ordering and the various functions are given by

$$\begin{aligned} \mathcal{H}_0 &= \frac{N\lambda_0^2}{4} I_0^2 \qquad with \qquad I_0 = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \\ e_0(q) &= \omega_q + \Delta_0(q), \qquad\qquad \Delta_0(q) = \frac{\lambda_0^2}{2\omega_q} I_0. \end{aligned} \tag{29}$$

The normal-ordered pion fields can be explicitly written as

$$: \vec{\pi}^2(\mathbf{x}) := \int \frac{d^3q_1 d^3q_2 e^{i(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}}{(2\pi)^3\sqrt{4\omega_1\omega_2}} \left[\left(A^+\sqrt{N + A^+A}\right)_{1,2} + \left(\sqrt{N + A^+A}A\right)_{-1,-2} + 2\left(A^+A\right)_{1,-2} \right]. \tag{30}$$

Clearly the Hamiltonian is an infinite series in powers of the new bosons as a direct result of the Holstein-Primakoff mapping. It then takes the form

$$H = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \mathcal{H}^{(3)} + \mathcal{H}^{(4)} + ... \tag{31}$$

The superscript in each term designates the power in the boson operators and these are either the sigma bosons b_q, b_q^+ or the newly introduced pion-pair bosons A, A^+. In fact, in addition to this expansion in the power of bosons there exists an underlying expansion in the square root of the number of pion charges, $\sqrt{N}$. The latter is particularly suitable for a perturbative treatment of the dynamics generated by the Hamiltonian eq. (28). The fact that N is arbitrary insures the preservation of the symmetries of H at each order. The perturbative calculation of all higher orders is performed in a selfconsistent basis which we will establish first. Since there are two bosons which are able to condense in the vacuum, the natural choice for the trial wave function of the ground state is a coherent state of the form

$$|\psi\rangle = exp\left[\sum_q \hat{d}_q\, A^+_{q,-q} + \langle b_0\rangle\, b_0^+ \right] |0\rangle. \tag{32}$$

To fix the parameters $\hat{d}_q$ and $\langle b_0\rangle$ one needs to apply the Ritz variational principle to the expectation value $\mathcal{H}^0 = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$. The latter can be evaluated and gives

$$\mathcal{H}^0 = (2\pi)^3\, N \left[\int \frac{d^3\mathbf{q}}{(2\pi)^3} \omega_q \frac{\hat{d}_q^2}{N} + \frac{\lambda_0^2\langle\hat{\sigma}\rangle^2}{2N}(I_0 + K_0) + \frac{\lambda_0^2}{4}(I_0 + K_0)^2 + \frac{\lambda_0^2\langle\hat{\sigma}\rangle^4}{4N^2} + \frac{\mu_0^2\langle\hat{\sigma}\rangle^2}{2N} - \frac{c\langle\hat{\sigma}\rangle}{\sqrt{N}} \right] \tag{33}$$

The parameter $\langle\hat{\sigma}\rangle = \frac{\langle b_0\rangle + \langle b_0^+\rangle}{\sqrt{(2\pi)^3 2\mathcal{E}_\sigma}}$, as defined in eq. (5), is nothing but the expectation value of the sigma field in the coherent state $|\psi\rangle$ while K_0 is given by

$$K_0 \;=\; \frac{1}{N}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{\left(\hat{d}_q \,+\, \sqrt{N \,+\, \hat{d}_q^2}\right)^2 \,-\, N}{2\omega_q}. \tag{34}$$

The minimum of H^0 determines the values of $\hat{d}_q$ and $s \;=\; \langle\hat{\sigma}\rangle/\sqrt{N}$, as solutions of the following equations

$$\begin{aligned} e_\pi(q)\left(2\,\hat{d}_q\,\sqrt{N \,+\, \hat{d}_q^2}\right) \quad &+ \quad \Delta_\pi(q)\left(2\,\hat{d}_q^2 \,+\, N\right) \;=\; 0 \\ \mu_0^2 \quad &+ \quad 2\,\omega_q\,\Delta_\pi(q) \;=\; \frac{c}{s} \end{aligned} \tag{35}$$

where $\Delta_\pi(q)$ and $e_\pi(q)$ are given by

$$e_\pi(q) \;=\; \omega_q \,+\, \Delta_\pi(q), \qquad\qquad \Delta_\pi(q) \;=\; \frac{\lambda_0^2}{2\,\omega_q}\left[I_0 \,+\, K_0 \,+\, s^2\right]. \tag{36}$$

The first equation in (35) can be reexpressed in the more suggestive form

$$\hat{d}_q^2 \;=\; \frac{N}{2}\left[\frac{e_\pi(q)}{\sqrt{e_\pi^2(q) \,-\, \Delta_\pi^2(q)}} \,-\, 1\right]. \tag{37}$$

From the canonical form of this 'gap equation' one can deduce the pion quasiparticle energy $\mathcal{E}_\pi(q)$ as

$$\mathcal{E}_\pi(q) = \sqrt{e_\pi^2(q) \,-\, \Delta_\pi^2(q)} \;=\; \sqrt{q^2 \,+\, \mu_0^2 \,+\, \lambda_0^2\,[I_0 \,+\, K_0 \,+\, s^2]}. \tag{38}$$

One can also deduce from eq. (36) that the shift $\hat{d}_q$ for the pion pairs can have a simple scaling, namely as $\sqrt{N}$, which shows the physical similarity of this parameter to the scalar field condensate $\langle\hat{\sigma}\rangle$. In what follows we will use a rescaled shift parameter

$$d_q \;=\; \hat{d}_q/\sqrt{N}.$$

An interesting identity which allows to transmit the selfconsistency of the gap equation in eq. (36) to the pion quasiparticle mass reads

$$\mathcal{E}_\pi(q) = \left(d_q \,-\, \sqrt{1 \,+\, d_q^2}\right)^2\,\omega_q \tag{39}$$

from which one can finally extract the Hartree-Bogoliubov (HB) mass of the pion,

$$m_\pi = \mathcal{E}_\pi(0),$$

and the value of the condensate via the two coupled equations

$$\begin{aligned} m_\pi^2 &= \mu_0^2 + \lambda_0^2 \left[I_\pi + s^2 \right] \\ \frac{c}{s} &= \mu_0^2 + \lambda_0^2 \left[I_\pi + s^2 \right] \end{aligned} \tag{40}$$

where $I_\pi = I_0 + K_0$ denotes the quasi-pion tadpole.

Having fixed the quasiparticle basis we can now expand the Hamiltonian in eq. (28) in powers of the shifted bosons operators $\tilde{A}_{q,p}$ and β_q defined as

$$\begin{aligned} \tilde{A}_{q,p} \,=\, A_{q,p} \quad &- \quad \sqrt{N} d_q \delta(q+p) \qquad\qquad \beta_q \,=\, b_q \,-\, \langle b_0 \rangle\, \delta(q) \\ with \qquad\qquad \tilde{A}|\psi\rangle \quad &= \quad \beta|\psi\rangle \,=\, 0. \end{aligned} \tag{41}$$

Knowing the scaling of all parameters in the Hamiltonian the latter can therefore be expanded without ambiguity according to

$$H \;=\; N\, H^{(0)} \,+\, \sqrt{N}\, H^{(1)} \,+\, H^{(2)} \,+\, \frac{1}{\sqrt{N}}\, H^{(3)} \,+\, \frac{1}{N}\, H^{(4)} \,+ ... \tag{42}$$

where each order is obtained by expanding the square roots in eq. (23). Using the parameter differentiation techniques of operators (see appendix 1) one can write down the contributions to the Hamiltonian to leading orders in $1/N$, namely $H^{(1)}$ and $H^{(2)}$:

$$\begin{aligned} H^{(1)} \quad &= \quad \sqrt{\frac{(2\pi)^3}{2\mathcal{E}_\sigma}} \, \left[\lambda_0^2 I_\pi s \,+\, \lambda_0^2 s^3 \,+\, \mu_0^2 s \,-\, c \right] \left(\beta_0 \,+\, \beta_0^+ \right) \\ &+ \qquad\quad \int d^3q \left[\omega_q d_q \,+\, \frac{\Delta_\pi(q) \left(d_q \,+\, \sqrt{1\,+\,d_q^2} \right)^2}{2\sqrt{1\,+\,d_q^2}} \right] \left(\tilde{A}^+_{q,-q} \,+\, \tilde{A}_{-q,q} \right) \end{aligned} \tag{43}$$

$$\begin{aligned} H^{(2)} \quad &= \quad \int d^3q \mathcal{E}_\sigma(q) \beta_q^+ \beta_q \quad + \quad \int d^3q \mathcal{E}_\pi(q) \left(1 \,+\, d_q^2 \right) \left(\tilde{A}^+ \tilde{A} \right)_{q,q} \\ &+ \quad \int d^3q d^3p \left[\frac{\Delta_\pi(q)}{2} \frac{\sqrt{1+d_q^2} - \sqrt{1+d_p^2}}{d_q^2 - d_p^2} \left(\left(d\tilde{A} + \tilde{A}^+ d \right)_{q,p} \tilde{A}_{p,q} \,+\, \tilde{A}^+_{q,p} \left(d\tilde{A} + \tilde{A}^+ d \right)_{p,q} \right) \right. \\ &+ \quad \left. \frac{d_q\, \Delta_\pi(q)}{d_q^2 \,-\, d_p^2} \left(\frac{1}{2\sqrt{1+d_q^2}} - \frac{\sqrt{1+d_q^2} - \sqrt{1+d_p^2}}{d_q^2 - d_p^2} \right) \left(d\tilde{A} + \tilde{A}^+ d \right)_{q,p} \left(d\tilde{A} + \tilde{A}^+ d \right)_{p,q} \right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \lambda_0^2 \, s \int \frac{d^3q_1 d^3q_2 d^3q_3}{\sqrt{(2\pi)^3 \, 8\mathcal{E}_\sigma(q_3)\omega(q_1)\omega(q_2)}} \, \delta^3(q_1+q_2+q_3)\Gamma_{1,2} \left[\beta_3 \,+\, \beta^+_{-3}\right] \left[\tilde{A}_{1,2} \,+\, \tilde{A}^+_{-1,-2}\right] \\
&+ \quad \frac{\lambda_0^2}{4} \int \frac{d^3q_1 d^3q_2 d^3q_3 d^3q_4 \quad \delta^3(q_1+q_2+q_3+q_4)}{(2\pi)^3\sqrt{16\omega(q_1)\omega(q_2)\omega(q_3)\omega(q_4)}} \, \Gamma_{1,2}\,\Gamma_{3,4} \left[\tilde{A}_{1,2} + \tilde{A}^+_{-1,-2}\right] \left[\tilde{A}_{3,4} + \tilde{A}^+_{-3,-4}\right]
\end{aligned}
\tag{44}
$$

with the following definitions for the sigma quasiparticle mass $\mathcal{E}_\sigma$ and for the factors $\Gamma_{i,j}$:

$$
\begin{aligned}
\mathcal{E}_\sigma^2 &= m_\pi^2 \,+\, 2\lambda_0^2 s^2 \\
\Gamma_{i,j} &= \frac{\left(d_i + \sqrt{1+d_i^2}\right)^2 - \left(d_j + \sqrt{1+d_j^2}\right)^2}{2\,(d_i \,-\, d_j)} .
\end{aligned}
\tag{45}
$$

The gap equation which we have derived in eq. (35) eliminates the linear part of the Hamiltonian, *i.e* $H^{(1)} \,=\, 0$ thus defining the HB basis. The leading contribution in the expansion is therefore of order one. As we have seen, an appropriate mapping of the boson pair operators into ideal bose operators automatically produces the correct result.

The Hamiltonian above can be recast in a more compact form which is particularly suitable for the forthcoming considerations. To achieve this a set of new operators $B^+_{q,p}, B_{q,p}$ is defined which corresponds to a rotation of the set of $\tilde{A}^+_{q,p}, \tilde{A}_{q,p}$ operators, according to

$$
B^+_{q,p} \,=\, G_{q,p}\,\tilde{A}^+_{q,p} \;-\; H_{q,p}\,\tilde{A}_{q,p}
\tag{46}
$$

In order for the $B^+_{q,p}, B_{q,p}$ operators to obey the same algebra as the $\tilde{A}^+_{q,p}, \tilde{A}_{q,p}$, the coefficients $G_{q,p}$ and $H_{q,p}$ have to be chosen such that

$$
G^2_{q,p} \;-\; H^2_{q,p} \,=\, 1.
\tag{47}
$$

Explicitly the functions $G_{q,p}$ and $H_{q,p}$ are given by

$$
G_{q,p} \pm H_{q,p} = \frac{d_q\sqrt{1\,+\,d_p^2} \,\pm\, d_p\sqrt{1\,+\,d_q^2}}{d_q \,\pm\, d_p} .
\tag{48}
$$

With these definitions one verifies that the symmetry of the operators $\tilde{A}^+_{q,p}, \tilde{A}_{q,p}$ with respect to the

interchange of the indices is also obeyed by the new bosons $B^+_{q,p}, B_{q,p}$, since $G_{q,p}$ and $H_{q,p}$ are symmetric under the interchange of the indices. The quadratic part of H now reads

$$
\begin{aligned}
H^{(2)} &= \int d^3q\ \mathcal{E}_\sigma(q)\ \beta^+_q\beta_q \quad + \quad \int d^3q d^3p\ \mathcal{E}_\pi(q)\ B^+_{q,p}B_{q,p} \\
&+ \lambda_0^2\, s \int \frac{d^3q_1 d^3q_2 d^3q_3 \quad \delta^3(q_1+q_2+q_3)}{\sqrt{(2\pi)^3\, 8\mathcal{E}_\sigma(q_3)\mathcal{E}_\pi(q_1)\mathcal{E}_\pi(q_2)}} \left[\beta_3 \,+\, \beta^+_{-3}\right]\left[B_{1,2} \,+\, B^+_{-1,-2}\right] \\
&+ \frac{\lambda_0^2}{4}\int \frac{d^3q_1 d^3q_2 d^3q_3 d^3q_4 \quad \delta^3(q_1+q_2+q_3+q_4)}{(2\pi)^3\sqrt{16\mathcal{E}_\pi(q_1)\mathcal{E}_\pi(q_2)\mathcal{E}_\pi(q_3)\mathcal{E}_\pi(q_4)}} \left[B_{1,2} \,+\, B^+_{-1,-2}\right]\left[B_{3,4} \,+\, B^+_{-3,-4}\right].
\end{aligned}
\tag{49}
$$

The next task is to diagonalize each order of the Hamiltonian. Parts with odd powers in the operators have to be eliminated by this procedure, as was the case for $H^{(1)}$. Expressing $H^{(2)}$ in diagonal form leads to an expression with uncoupled oscillators as will be seen below.

The diagonalization of $H^{(2)}$ can be performed by using a general Bogoliubov rotation which mixes the configuration of the quasisigma Boson and the bosonized pair of pions according to

$$
Q^+_\nu(\vec{p}) \,=\, U^{(1)}_\nu(\vec{p})\beta^+_{\vec{p}} \,-\, V^{(1)}_\nu(\vec{p})\beta_{-\vec{p}} \,+\, \frac{1}{\sqrt{2}}\sum_q \left[U^{(2)}_\nu(\vec{p},\vec{q})B^+_{\vec{q},\vec{p}-\vec{q}} \,-\, V^{(2)}_\nu(\vec{p},\vec{q})B_{-\vec{q},-\vec{p}+\vec{q}}\right]. \tag{50}
$$

The operator Q^+_ν can be considered as an RPA excitation operator. The rotation in the RPA operator in eq. (50) is performed under the kinematical constraint $\delta(\vec{q}_1+\vec{q}_2-\vec{p})$, where $\vec{p}$ denotes the total momentum.

Using Rowe's equation of motion [17]

$$
\langle RPA|\left[\delta Q_\nu(\vec{p})\ ,\ \left[H^{(2)}\ ,\ Q^+_\nu(\vec{p})\right]\right]|RPA\rangle = \Omega_\nu(\vec{p})\langle RPA|\left[\delta Q_\nu(\vec{p})\ ,\ Q^+_\nu(\vec{p})\right]|RPA\rangle \tag{51}
$$

and keeping in eq. (51) the full RPA ground state defined by $Q_\nu|RPA\rangle = 0$ would lead to a selfconsistent form of the RPA equations (see [23]) if higher than second-order terms are included in the expansion of H. Retaining only $H^{(2)}$ linearizes the equation of motion and one obtains the familiar form of the RPA equations

$$
\int d^3\vec{q}_2 \begin{pmatrix} \mathcal{A}_{\vec{p}}(\vec{q}_1,\vec{q}_2) & \mathcal{B}_{\vec{p}}(\vec{q}_1,\vec{q}_2) \\ \mathcal{B}_{\vec{p}}(\vec{q}_1,\vec{q}_2) & \mathcal{A}_{\vec{p}}(\vec{q}_1,\vec{q}_2) \end{pmatrix} \begin{pmatrix} \mathcal{U}_\nu(\vec{p},\vec{q}_2) \\ \mathcal{V}_\nu(\vec{p},\vec{q}_2) \end{pmatrix} \,=\, \Omega_\nu(\vec{p})\mathcal{N} \begin{pmatrix} \mathcal{U}_\nu(\vec{p},\vec{q}_1) \\ \mathcal{V}_\nu(\vec{p},\vec{q}_1) \end{pmatrix} \tag{52}
$$

where $\mathcal{A}_{\vec{p}}$, $\mathcal{B}_{\vec{p}}$ are 2×2 matrices given by

$$
\mathcal{A}_{\vec{p}}(\vec{q}_1,\vec{q}_2) = \begin{pmatrix} \mathcal{E}_\sigma(\vec{p}) & 0 \\ 0 & [\mathcal{E}_\pi(\vec{q}_1)+\mathcal{E}_\pi(\vec{p}-\vec{q}_1)] \end{pmatrix} \delta(\vec{q}_1-\vec{q}_2) \quad + \quad \mathcal{B}_{\vec{p}}(\vec{q}_1,\vec{q}_2) \tag{53}
$$

with

$$\mathcal{B}_{\vec{p}}(\vec{q}_1,\vec{q}_2) = \begin{pmatrix} 0 & \frac{\sqrt{2}\lambda_0^2 s}{\sqrt{(2\pi)^3 8\mathcal{E}_\sigma(\vec{p})\mathcal{E}_\pi(\vec{q}_2)\mathcal{E}_\pi(\vec{p}-\vec{q}_2)}} \\ \frac{\sqrt{2}\lambda_0^2 s}{\sqrt{(2\pi)^3 8\mathcal{E}_\sigma(\vec{p})\mathcal{E}_\pi(\vec{q}_1)\mathcal{E}_\pi(\vec{p}-\vec{q}_1)}} & \frac{\lambda_0^2}{(2\pi)^3\sqrt{16\mathcal{E}_\pi(\vec{q}_1)\mathcal{E}_\pi(\vec{q}_2)\mathcal{E}_\pi(\vec{p}-\vec{q}_1)\mathcal{E}_\pi(\vec{p}-\vec{q}_2)}} \end{pmatrix} \tag{54}$$

respectively. The 4×4 norm matrix $\mathcal{N}$ as well as the two-component vectors $\mathcal{U}_\nu$, $\mathcal{V}_\nu$ are given by

$$\begin{aligned} \mathcal{N} &= \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix} \qquad I_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathcal{U}_\nu(\vec{p},\vec{q}) &= \begin{pmatrix} U_\nu^{(1)}(\vec{p}) \\ U_\nu^{(2)}(\vec{p},\vec{q}) \end{pmatrix} \qquad \mathcal{V}_\nu(\vec{p},\vec{q}) = \begin{pmatrix} V_\nu^{(1)}(\vec{p}) \\ V_\nu^{(2)}(\vec{p},\vec{q}) \end{pmatrix}. \end{aligned} \tag{55}$$

One can now solve the eigenvalue problem and extract the RPA frequencies to obtain

$$\Omega_\nu^2(\vec{p}) \;=\; \mathcal{E}_\sigma^2 + \frac{2\lambda_0^4 s^2\,\Sigma_{\pi\pi}(\Omega_\nu^2(\vec{p}))}{1 \,-\, \lambda_0^2\Sigma_{\pi\pi}(\Omega_\nu^2(\vec{p}))} + \vec{p}^2 \tag{56}$$

with $\Sigma_{\pi\pi}(\Omega_\nu^2(\vec{p}))$ being the Lorentz invariant $\pi\pi$ self energy of the sigma given by

$$\Sigma_{\pi\pi}(\Omega_\nu^2(\vec{p})) \;=\; \int \frac{d^3q}{(2\pi)^3}\, \frac{\mathcal{E}_\pi(q) \,+\, \mathcal{E}_\pi(p-q)}{2\,\mathcal{E}_\pi(q)\mathcal{E}_\pi(p-q)}\, \frac{1}{\Omega_\nu^2(p) \,-\, (\mathcal{E}_\pi(q) \,+\, \mathcal{E}_\pi(p-q))^2}. \tag{57}$$

It should be noted that, in contrast to (16), where the quasipion was massive in the chiral limit, here the $\Sigma_{\pi\pi}$ self energy in (57) is built on massless Goldstone pions. Using the usual orthonormalisation condition of the RPA states, *i.e* $\langle RPA|Q_\nu Q_{\nu'}^+|RPA\rangle = \delta_{\nu\nu'}$, one obtains the following solution for the RPA eigenvectors

$$\begin{aligned} U_\nu^{(1)}(\vec{p}) &= \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2}} \frac{V_{\pi\pi\to\sigma}}{\Xi(\vec{p},\Omega_\nu)} \frac{1}{\sqrt{2\mathcal{E}_\sigma(\vec{p})}} \frac{1}{\Omega_\nu - \mathcal{E}_\sigma(\vec{p})} \\ V_\nu^{(1)}(\vec{p}) &= -\frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2}} \frac{V_{\pi\pi\to\sigma}}{\Xi(\vec{p},\Omega_\nu)} \frac{1}{\sqrt{2\mathcal{E}_\sigma(\vec{p})}} \frac{1}{\Omega_\nu + \mathcal{E}_\sigma(\vec{p})} \\ U_\nu^{(2)}(\vec{p},\vec{q}) &= \frac{1}{2} \frac{V_{\pi\pi\to\pi\pi}}{\Xi(\vec{p},\Omega_\nu)} \frac{1}{\sqrt{4\mathcal{E}_\pi(\vec{q})\mathcal{E}_\pi(\vec{p}-\vec{q})}} \frac{1}{\Omega_\nu - \mathcal{E}_\pi(\vec{q}) - \mathcal{E}_\pi(\vec{p}-\vec{q})} \\ V_\nu^{(2)}(\vec{p},\vec{q}) &= -\frac{1}{2} \frac{V_{\pi\pi\to\pi\pi}}{\Xi(\vec{p},\Omega_\nu)} \frac{1}{\sqrt{4\mathcal{E}_\pi(\vec{q})\mathcal{E}_\pi(\vec{p}-\vec{q})}} \frac{1}{\Omega_\nu + \mathcal{E}_\pi(\vec{q}) + \mathcal{E}_\pi(\vec{p}-\vec{q})} \end{aligned} \tag{58}$$

where $V_{\pi\pi\to\sigma}$ and $V_{\pi\pi\to\pi\pi}(\Omega_\pi,\vec{p})$ stand for the $\pi\pi\to\sigma$ and $\pi\pi\to\pi\pi$ tree level transition matrix respectively. These as well as the function Ξ are given by

$$\Xi(\Omega_\nu, \vec{p}) \;=\; (2\pi)^{3/2} \left[\frac{\Omega_\nu \; (V_{\pi\pi\to\sigma})^2}{\left[\Omega_\pi^2 - \mathcal{E}_\sigma^2(\vec{p})\right]^2} \;-\; \frac{1}{4} \left(V_{\pi\pi\to\pi\pi}(\Omega_\nu, \vec{p})\right)^2 \frac{\partial \Sigma_{\pi\pi}(\Omega_\nu, \vec{p})}{\partial \Omega_\nu} \right]^{1/2}$$
$$V_{\pi\pi\to\pi\pi}(\Omega_\nu, \vec{p}) \;=\; 2\lambda_0^2 \frac{\Omega_\nu^2 - \mathcal{E}_\pi^2(\vec{p})}{\Omega_\nu^2 - \mathcal{E}_\sigma^2(\vec{p})}, \qquad V_{\pi\pi\to\sigma} \;=\; 2\lambda_0^2 s. \tag{59}$$

This concludes the discussion of the leading order dynamics. The physical observables, at this order, are given by the expressions in eq. (40) for the pion mass and the condensate and by eq. (56) for the sigma mass. These were the same results obtained at the end of the previous section.

In the next section some remaining subtleties in relation with the properties of the RPA approximation will be pointed out.

4 The $\pi - \pi$ Goldstone Mode and its Dynamics, Sum Rules.

In the chiral limit the RPA solutions have to yield a spurious mode corresponding physically to the case of two non-interacting Goldstone pions. This mode has to occur if the massless pions have zero total and zero relative momentum. Such a solution is not manifest from expression (56) which has been obtained by solving the RPA equations in the sigma channel via Feshbach projection. In the chiral limit and for vanishing total momentum $\vec{p}$ the RPA frequency will just become the sigma mass as to be expected and as can also be seen by comparing eq. (56) with eq. (20). That a Goldstone mode (*i.e.* a Goldstone mode for the two pions) must also exist, can be deduced from the following symmetry considerations (see also section 5).

Since, without explicit symmetry breaking, the sigma-model is invariant under chiral transformations, the Hamiltonian must fulfill the following commutation relations

$$\left[H\,,\,Q_5^i\right] = \left[H\,,\,\vec{Q}_5\vec{Q}_5\right] = 0 \tag{60}$$

since Q_5^i, the i-th axial charge, is a generator of the symmetry. The second relation is most interesting in the present context since it involves two-pion degrees of freedom. As for single Q_5^i operators, the product of two axial charges can also be mapped into the bosons $\tilde{A}^+, \tilde{A}$ or equivalently B^+, B. This allows to express $S \equiv \vec{Q}_5\vec{Q}_5$ in an infinite series in powers of the new bosons

$$S \;=\; N^2\, S^{(0)} \;+\; N^{3/2}\, S^{(1)} \;+\; N\, S^{(2)} \;+\; \sqrt{N}\, S^{(3)} \;+\; S^{(4)} \;+\; \frac{1}{\sqrt{N}}\, S^{(5)} \;+ \ldots \tag{61}$$

The $S^{(p)}$ are organized according to the power p of the bosons β, β^+, B, B^+. It is important to restate

that, as in the case of the Hamiltonian expansion in eqs. (42,43,44), no normal ordering is assumed. Therefore a given term $S^{(p)}$ may very well contain $p-1, p-2, ..$ powers of the bosons. The real expansion parameter in eq. (42) as well as in eq. (61) is $\sqrt{N}$.
The commutation relation eq. (60) will thus read

$$[H\,,\,S] = \sum_p N^{3-\frac{p}{2}}\,C^{(p-1)} \,=\, 0. \tag{62}$$

According to the basic observation that the polynomial is zero if and only if each of the monoms is vanishing, one can thus extract less stringent commutation relations between parts of the total Hamiltonian H and the symmetry operator S. These are simply

$$C^{(p)} \;=\; \left[H^{(1)}\,,\,S^{(p+1)}\right] + \left[H^{(2)}\,,\,S^{(p)}\right] + \ldots . + \left[H^{(p+1)}\,,\,S^{(1)}\right] \;=\; 0, \qquad\qquad \forall\, p \geq 0. \tag{63}$$

The relevant commutation relation to the order we are working is given by $C^{(1)} \;=\; 0$ and reads explicitly

$$\left[H^{(1)}\,,\,S^{(2)}\right] + \left[H^{(2)}\,,\,S^{(1)}\right] \;=\; 0. \tag{64}$$

Recalling the fact that $H^{(1)}$ is eliminated by the construction of the variational (HB) basis, one is left only with

$$\left[H^{(2)}\,,\,S^{(1)}\right] \;=\; 0. \tag{65}$$

This proves that a zero mode must exist. For the discussion below it is useful to give an explicit form of the symmetry operator. Since $S^{(1)}$ contains only unit powers of the bosons β, β^{+}, B, B^{+}, the RPA operator Eq. (50) is therefore the most general ansatz with single boson excitations containing $S^{(1)}$. Explicitly, $S^{(1)}$ can be constructed as

$$S^{(1)} \;=\; \left[U^{(1)}_{sp}(\vec{0})\beta^{+}_{0} \;-\; V^{(1)}_{sp}(\vec{0})\beta_0\right] \quad + \quad \left[U^{(2)}_{sp}(\vec{0},\vec{0})B^{+}_{00} \;-\; V^{(2)}_{sp}(\vec{0},\vec{0})B_{00}\right] \tag{66}$$

with

$$\mathcal{U}_{sp}(\vec{0},\vec{0}) \;=\; -\,\mathcal{V}_{sp}(\vec{0},\vec{0}) \;=\; \begin{pmatrix} (2\pi)^{\frac{3}{2}}\,s\,(2\,\mathcal{E}_{\sigma}(0))^{-\frac{1}{2}}\,\mathcal{E}_{\pi}(0) \\ -(2\pi)^{3}\,\frac{s^2}{2}\mathcal{E}_{\pi}(0) \end{pmatrix}. \tag{67}$$

Physically this operator creates a zero-momentum sigma meson and a pion pair of vanishing total- as well as relative momentum.

As mentioned above, it is instructive to express $H^{(2)}$ in the RPA basis. We shall follow the usual steps that can be found in the literature (see for instance [7]). First it is easy to see that, by introducing

the RPA matrix, $H^{(2)}$ can be written in matrix form as

$$H^{(2)} = -\frac{1}{2}Tr\,(\mathcal{A}-\mathcal{B}) \ + \ \frac{1}{2} \ \left(\beta^+ \ \frac{1}{\sqrt{2}}B^+ \ \beta \ \frac{1}{\sqrt{2}}B\right) \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \beta \\ \frac{1}{\sqrt{2}}B \\ \beta^+ \\ \frac{1}{\sqrt{2}}B^+ \end{pmatrix} . \tag{68}$$

Following [7] we introduce a matrix Υ and denote the RPA matrix by $\mathcal{S}$:

$$\Upsilon = \begin{pmatrix} \mathcal{U} & \mathcal{V}^* \\ \mathcal{V} & \mathcal{U}^* \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}. \tag{69}$$

Then the closure relation of the RPA basis [7] reads

$$\Upsilon \mathcal{N} \Upsilon^+ = \overline{\mathcal{N}} \tag{70}$$

where $\overline{\mathcal{N}}$ is a two by two norme matrice defined by substituting 1 to I_d in $\mathcal{N}$ of eq.(55). This allows to reexpress the original boson operators in terms of RPA excitation operators as

$$\begin{aligned}
\beta^+_p &= \sum_{\nu>0} \left(U^{(1)\,*}_\nu(\vec{p})Q^+_\nu(\vec{p}) \ + \ V^{(1)}_\nu(\vec{p})Q_\nu(-\vec{p})\right) \\
\beta_p &= \sum_{\nu>0} \left(V^{(1)\,*}_\nu(\vec{p})Q^+_\nu(-\vec{p}) \ + \ U^{(1)}_\nu(\vec{p})Q_\nu(\vec{p})\right) \\
B^+_{q,p} &= \sqrt{2}\sum_{\nu>0} \left(U^{(2)\,*}_\nu(\vec{q},\vec{p})Q^+_\nu(\vec{p}+\vec{q}) \ + \ V^{(2)}_\nu(-\vec{q},-\vec{p})Q_\nu(-\vec{p}-\vec{q})\right) \\
B_{q,p} &= \sqrt{2}\sum_{\nu>0} \left(V^{(2)\,*}_\nu(-\vec{q},-\vec{p})Q^+_\nu(-\vec{p}-\vec{q}) \ + \ U^{(2)}_\nu(\vec{q},\vec{p})Q_\nu(\vec{p}+\vec{q})\right).
\end{aligned} \tag{71}$$

Now, using the orthonormalisation condition of the RPA basis which in compact notation reads

$$\Upsilon^+ \overline{\mathcal{N}} \Upsilon = \mathcal{N} \tag{72}$$

one can extract, with the help of the RPA equation, the components of the eigenvectors $\mathcal{U}_\nu(\vec{p},\vec{q})$ and $\mathcal{V}_\nu(\vec{p},\vec{q})$ as defined in eq. (55) and explicitly given in eq. (58). Finally, with the RPA excitation operators as defined in eq. (50) the Hamiltonian takes the following form

$$H^{(2)} = -\frac{1}{2}Tr\,(\mathcal{A}-\mathcal{B}) \ + \ \frac{1}{2} \ (Q^+ \ Q)\,\Upsilon^+\mathcal{S}\Upsilon \begin{pmatrix} Q \\ Q^+ \end{pmatrix}. \tag{73}$$

In the exact chiral limit we have seen that a zero mode (with $\Omega_{sp} = 0$) is present in the set of RPA

solutions which requires special treatment when inverting the RPA. The procedure is standard and can be found in textbooks (see for instance [7]). Taking into account the zero mode which appears in the chiral limit, the completeness of the RPA basis is again recovered and the Hamiltonian can be expressed as follows

$$H^{(2)} = -\frac{1}{2} Tr\,(\mathcal{A} - \mathcal{B}) \, + \, \frac{1}{2} \sum_{\vec{p},\, \nu > 0} \Omega_\nu(\vec{p}) \quad + \sum_{\vec{p},\, \nu > 0} \Omega_\nu(\vec{p})\, Q_\nu^+(\vec{p})\, Q_\nu(\vec{p}) \quad + \quad \frac{\left(S^{(1)}\right)^2}{2\,\mathcal{I}_0 V}. \tag{74}$$

$S^{(1)}$ is the $\mathcal{O}(1)$ symmetry generator given in eq. (61) and $\mathcal{I}_0$ is the moment of inertia per unit volume (with V the volume of the system) which can be calculated using the Valatin-Thouless equations [24]:

$$\begin{aligned} \left[H^{(2)} \,,\, C^{(1)}\right] &= S^{(1)} \\ \left[S^{(1)} \,,\, C^{(1)}\right] &= \mathcal{I}_0 \end{aligned} \tag{75}$$

The operator $C^{(1)}$ is an anti-Hermitian operator proportional to the canonical variable $T^{(1)}$, conjugate to the zero mode induced by $S^{(1)}$ such that

$$C^{(1)} = i\mathcal{I}_0 T^{(1)} \qquad \text{with} \quad \left[T^{(1)} \,,\, S^{(1)}\right] = i \tag{76}$$

The Valatin-Thouless equations in eq. (75) are sufficient to determine both $T^{(1)}$ and the moment of Inertia $\mathcal{I}_0$/ unit volume. An explicit calculation gives

$$\begin{aligned} T^{(1)} &= \frac{i}{(2\pi)^3 2 s^2 m_\pi \, + \, 1} \left[\left(B_{00}^+ \, - \, B_{00} \right) \, - \, \frac{\sqrt{(2\pi)^3 2\mathcal{E}_\sigma}}{(2\pi)^3 2 s\, m_\pi} \left(\beta_0^+ \, - \, \beta_0 \right) \right] \\ \mathcal{I}_0 &= \frac{s^2}{4} \end{aligned} \tag{77}$$

The expression for the moment of inertia suggests the very intuitive picture that the inertia of the vacuum against a chiral rotation is simply given by the amount of the spontaneous breaking via the condensate, s. Of course in general the volume V in eq. (74) is infinite and thus the kinetic energy term of chiral rotation in (74) is zero. However, in relativistic heavy ion collisions, situations may occur where there are small droplets of pionic matter with spontaneously broken chiral symmetry, surrounded by regions in the symmetry restored phase. In such cases the inertia is finite and we must keep the corresponding term.

Before closing this section, we wish to address briefly the question of the energy-weighted sum rule (EWSR). For any hermitian operator, F, this sum rule, to leading order in the $1/N$-expansion, is given

by

$$m_F^1 = \sum_{\nu>0} \Omega_\nu |\langle \nu | F | RPA \rangle|^2 = \frac{1}{2} \langle \psi | [F, [H^{(2)}, F]] | \psi \rangle \tag{78}$$

where $|\psi\rangle$ is the coherent state specified in eq. (32). The proof goes along the standard arguments which can be found for instance in [7, 25].
Among other things such sum rules can serve as a test for the correctness of numerical calculations. For example, a particular simple result is obtained with $F(\vec{x}) = \hat{\sigma}(\vec{x})$ for which the EWSR gives $m_F^1(q) = 1$, where q is the three-momentum carried by the σ-field. Further sum rules with different excitation operators will be discussed in a forthcoming paper.

This concludes the formal discussion of the $1/N$ expansion and the properties of the ensuing RPA problem. We will now turn to the construction of the $\pi\pi$ T-matrix.

5 The $\pi\pi$-scattering equation

A $\pi\pi$-scattering equation can be deduced from eqs. (51, 52) by eliminating the sigma subspace with the help of a Feshbach projection. This procedure adds to the original $\pi\pi$-contact term of the Lagrangian an effective part which corresponds to s-channel sigma exchange. To leading order in the $1/N$-expansion the projected RPA equation is given by

$$\int d^3\vec{q}_1 \left([\mathcal{E}_\pi(\vec{q}_1) + \mathcal{E}_\pi(\vec{p}-\vec{q}_1)] \, \delta(\vec{q}_1 - \vec{q}_2) \, I_d + \Lambda(\Omega_\nu, \vec{p}, \vec{q}_1, \vec{q}_2) \right) \begin{pmatrix} U_\nu^{(2)}(\vec{p}, \vec{q}_1) \\ V_\nu^{(2)}(\vec{p}, \vec{q}_1) \end{pmatrix} = \Omega_\nu(\vec{p}) \begin{pmatrix} U_\nu^{(2)}(\vec{p}, \vec{q}_2) \\ -V_\nu^{(2)}(\vec{p}, \vec{q}_2) \end{pmatrix}. \tag{79}$$

Here I_d is the two by two identity matrix and the interaction matrix Λ is given by

$$\begin{aligned} \Lambda(\Omega_\nu, \vec{p}, \vec{q}_1, \vec{q}_2) &= \lambda_0^2 \frac{\Omega_\nu^2 - \mathcal{E}_\pi^2(\vec{p})}{\Omega_\nu^2 - \mathcal{E}_\sigma^2(\vec{p})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{d(\vec{p}, \vec{q}_1, \vec{q}_2)}{(2\pi)^3} \\ d(\vec{p}, \vec{q}_1, \vec{q}_2) &= [16 \mathcal{E}_\pi(\vec{q}_1) \mathcal{E}_\pi(\vec{q}_2) \mathcal{E}_\pi(\vec{p}-\vec{q}_1) \mathcal{E}_\pi(\vec{p}-\vec{q}_2)]^{-1/2} \end{aligned} \tag{80}$$

From eqs. (79) and (80) it is now straightforward to construct the corresponding two-pion propagator in the scalar-isoscalar channel:

$$G_{\pi\pi}(t-t', \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = -i \, \langle RPA | T \left(\vec{\pi}(\vec{x}_1) \vec{\pi}(\vec{x}_2) \right)_t \left(\vec{\pi}(\vec{x}_4) \vec{\pi}(\vec{x}_3) \right)_{t'} | RPA \rangle \tag{81}$$

where T denotes the time-ordering operator and $|RPA\rangle$ is the RPA-correlated vacuum. The Green's

function only depends on the time difference $t-t'$ but, in spite of being non-covariant, still contains the full dynamics at this order since t or u-channels only appear in next order. Taking the Fourier transform of $G_{\pi\pi}$ and keeping the relevant part to leading order of the expansion in the new bosons B, B^+ gives

$$G_{\pi\pi}(t-t', \vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) = N\, d(\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) \langle RPA | T \left(B_{q_1,q_2} + B^+_{-q_1,-q_2} \right)_t \left(B_{q_3,q_4} + B^+_{-q_3,-q_4} \right)_{t'} |RPA\rangle \tag{82}$$

The Fourier transform of the time difference $t-t'$ defines the center of mass energy E . At this order the two-pion Green's function is related in the usual way to the T-matrix

$$G_{\pi\pi}(E, \vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) = G^0_{\pi\pi}(E, \vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) + G^0_{\pi\pi}(E, \vec{q}_1, \vec{q}_2) T(E, \vec{p}) G^0_{\pi\pi}(E, \vec{q}_3, \vec{q}_4) \tag{83}$$

and a straightforward reduction then gives the T-matrix as

$$\begin{aligned} \int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \; G^{0\,-1}_{\pi\pi}(E, \vec{q}_1, \vec{q}_2, \vec{k}_1, \vec{k}_2) \quad & G_{\pi\pi}(E, \vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) \; G^{0\,-1}_{\pi\pi}(E, \vec{k}_3, \vec{k}_4, \vec{q}_3, \vec{q}_4) \\ = \quad & \tfrac{1}{(2\pi)^3}\, \delta(\vec{q}_1 + \vec{q}_2 - \vec{q}_3 - \vec{q}_4)\, T_{\pi\pi}(E, \vec{p}) \end{aligned} \tag{84}$$

where $\vec{p}$ is the total three-momentum of the pion pair, $\vec{p} = \vec{q}_1 + \vec{q}_2 = \vec{q}_3 + \vec{q}_4$. On the other hand $G^0_{\pi\pi}$ is the free two-pion Green's function defined as

$$\begin{aligned} G^0_{\pi\pi}(t-t', \vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) \quad &= \quad N\, d(\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4) \langle \psi | T \left(B_{q_1,q_2} + B^+_{-q_1,-q_2} \right)_t \left(B_{q_3,q_4} + B^+_{-q_3,-q_4} \right)_{t'} |\psi\rangle \\ &= \quad G^0_{\pi\pi}(t-t', \vec{q}_1, \vec{q}_2) \left[\delta(\vec{q}_1 - \vec{q}_3)\delta(\vec{q}_2 - \vec{q}_4) + \delta(\vec{q}_1 - \vec{q}_4)\delta(\vec{q}_2 - \vec{q}_3) \right] \end{aligned} \tag{85}$$

where $|\psi\rangle$ is the coherent state defined earlier. In energy space one has

$$G^0_{\pi\pi}(E, \vec{q}_1, \vec{q}_2) = \frac{\mathcal{E}_\pi(\vec{q}_1) + \mathcal{E}_\pi(\vec{q}_2)}{2\, \mathcal{E}_\pi(\vec{q}_1) \mathcal{E}_\pi(\vec{q}_2)} \; \frac{1}{E^2 - \left(\mathcal{E}_\pi(\vec{q}_1) + \mathcal{E}_\pi(\vec{q}_2) \right)^2 + i\eta} \tag{86}$$

Using the Heisenberg picture, the reduction in eq. (84) can be performed explicitly with the help of the RPA representation in eqs. (71, 74) as well as the RPA solutions in eqs. (56, 57, 58 , 59). This yields the following spectral representation for the T-matrix

$$T_{\pi\pi}(E, \vec{p}) \; = \; \frac{1}{2} \sum_{\nu > 0} \left| \frac{V_{\pi\pi}(\Omega_\nu, \vec{p})}{\Xi(\Omega_\nu, \vec{p})} \right|^2 \left(\frac{1}{E - \Omega_\nu(\vec{p}) + i\eta} - \frac{1}{E + \Omega_\nu(\vec{p}) - i\eta} \right) \tag{87}$$

where the $V_{\pi\pi}$ and Ξ functions have been given in eq. (59). An equivalent way of solving the scattering

problem in the RPA is to extract the bare $\pi\pi$-interaction $V_{\pi\pi}$ which serves as the kernel of a Lippmann-Schwinger equation:

$$T_{\pi\pi}(E,\vec{p}) \;=\; V_{\pi\pi}(E,\vec{p}) \,+\, \frac{1}{2}\int \frac{d\vec{q}}{(2\pi)^3}\, V_{\pi\pi}(E,\vec{p}) G^0_{\pi\pi}(E,\vec{p},\vec{q}) T_{\pi\pi}(E,\vec{p}) \tag{88}$$

with $G^0_{\pi\pi}$ as given in (86) and the bare interaction $V_{\pi\pi}$ obtained by reducing the Green's function in eq. (81) to lowest order in the dimensionless coupling constant, λ_0^2. The solution of the above equation has a simple algebraic form

$$\begin{aligned}
T(E,\vec{p}) \;&=\; \frac{V_{\pi\pi}(E,\vec{p})}{1 \,-\, \frac{1}{2}V_{\pi\pi}(E,\vec{p})\Sigma_{\pi\pi}(E,\vec{p})} \\
V_{\pi\pi}(E,\vec{p}) \;&=\; 2\lambda_0^2 \frac{E^2 \,-\, \mathcal{E}^2_\pi(\vec{p})}{E^2 \,-\, \mathcal{E}^2_\sigma(\vec{p})} \\
\lambda_0^2 \;&=\; N \frac{\mathcal{E}^2_\sigma(\vec{0}) - \mathcal{E}^2_\pi(\vec{0})}{2f_\pi^2} \qquad\quad \textit{with} \quad f_\pi^2 = Ns^2
\end{aligned} \tag{89}$$

where $\Sigma_{\pi\pi}(E,\vec{p})$ denotes the $\pi\pi$-bubble defined in eq. (57). The two forms of the T-matrix given above are equivalent. This can be verified by taking the residues at each pole of $T_{\pi\pi}$ from the expression in eq. (89) which then directly leads to the spectral representation in eq. (87). From eqs. (82, 83, 84, 89) and from the imaginary part of the total two-pion propagator $ImG_{\pi\pi}$ one deduces that the spectrum starts at $2m_\pi$ i.e. it is gapless in the chiral limit, in agreement with the Goldstone Theorem (see sect. 4). The σ -propagator can also be calculated starting from the related two-point Green's function

$$D_\sigma(t_1 - t_2, \vec{x}_1, \vec{x}_2) \;=\; -i\,\langle RPA | T\sigma(\vec{x}_1, t_1)\sigma(\vec{x}_2, t_2) | RPA\rangle \tag{90}$$

which, after Fourier transformation, can be written in spectral representation as

$$D_\sigma(E,\vec{p}) \;=\; \frac{1}{2}\sum_{\nu>0}\left|\frac{V_{\pi\pi\sigma}}{\Xi(\Omega_\nu,\vec{p})}\frac{1}{\Omega_\nu^2 - \mathcal{E}^2_\sigma(\vec{p})}\right|^2 \left(\frac{1}{E \,-\, \Omega_\nu(\vec{p}) \,+\, i\eta} \,-\, \frac{1}{E \,+\, \Omega_\nu(\vec{p}) \,-\, i\eta}\right). \tag{91}$$

Alternatively one can obtain D_σ by solving a Dyson equation with the mass operator containing the $\pi\pi$-RPA fluctuations. This leads to

$$D_\sigma(E,\vec{p}) \;=\; \left[E^2 \,-\, \mathcal{E}^2_\sigma(\vec{p}) \,-\, \frac{2\lambda_0^4 s^2\, \Sigma_{\pi\pi}(E,\vec{p})}{1 \,-\, \lambda_0^2 \Sigma_{\pi\pi}(E,\vec{p})}\right]^{-1} \tag{92}$$

and again these two forms of D_σ are equivalent, as can be seen by taking the residue at each pole in

eq. (92) which yields the spectral form (91). From these observations the scattering matrix $T_{\pi\pi}$ can be reexpressed as

$$T_{\pi\pi}(p^2) \;=\; \frac{D_\pi^{-1}(p^2) \;-\; D_\sigma^{-1}(p^2)}{s^2}\frac{D_\sigma(p^2)}{D_\pi(p^2)} \tag{93}$$

where $p^2 = E^2 - \vec{p}^2$. As we have seen above to this order in the $1/N$ expansion the pion is obtained in the Hartree approximation, and D_π is therefore given by

$$D_\pi(p^2) \;=\; \frac{1}{p^2 \;-\; m_\pi^2}. \tag{94}$$

Thus one pion can be taken soft (*i.e.* $m_\pi = 0$) in which case the above expression reduces to a Ward identity, linking the pion four-point function to the sigma and pion two-point functions [18]. The expression in (93) is more general, however, and includes the soft-pion limit as a special case. Finally it can be seen that, at the physical threshold $E^2 = 4m_\pi^2, \vec{p} = \vec{0}$, the T-matrix is directly proportional to the pion mass and hence vanishes in the chiral limit. This is, of course, required by the Goldstone nature of the pions. Away from the chiral limit it is interesting to compare the a_0^0 scattering length resulting from the T-matrix in eq. (89) with the well-known tree level results. To lowest order in the coupling constant and for $N = 3$, the a_0^0 scattering length can be read off from eq. (89) and yields

$$a^0_{0\,tree} \;=\; \frac{9}{32\pi}\frac{m_\pi}{f_\pi^2} \tag{95}$$

while the standard tree-level value is $a_0^0 = \frac{7}{32\pi}\frac{m_\pi}{f_\pi^2}$. The difference stems from the fact that, to leading-order in $1/N$, only s-channel contributions appear while the standard Weinberg tree-level result contains t- and u-channel pieces as well. These contributions are known to be repulsive and, in the $1/N$ expansion, they only enter in next-to-leading order. It is tempting to obtain the a_0^0 scattering length for the full T-matrix. Unfortunately for any reasonable choice of parameters (the cut-off, Λ, and the physical sigma mass, $\mathcal{E}_\sigma$) no acceptable fit of the scalar-isoscalar phase shifts can be obtained from eq. (89). This is to be expected since the RPA resummation of the scattering series will increase a_0^0 further from its tree level value, which is already too large. This hints to the fact that higher orders in the $1/N$ expansion will be crucial to get a quantitative description of the empirical $s-$wave phase shifts. It will be very interesting to work these out explicitly, exhibiting u- and t- channel contributions. This will be done in the future.

6 Summary and Outlook

In an attempt to obtain a symmetry-conserving $\pi\pi$-scattering equation, consistent with unitarity, we have investigated the cut-off version of the $O(N+1)$ linear sigma-model. Increasing interest in connection with

two-pion correlations in hadronic matter makes such studies particularly timely. On the one hand, as is well known, two-pion correlations play a crucial role in the N-N interaction and studies of the in-medium corrections are still in their infancy [2, 26]. On the other hand, in purely pionic matter, the $\pi\pi$-interaction, due to Bose statistics, may become strongly enhanced such that, in the scalar-isoscalar channel, two-pion bound states are likely to appear at higher temperature. Obviously, such features cannot be treated in chiral perturbation theory and non-perturbative schemes, respecting constraints from chiral symmetry, are called for. The present paper provides a novel approach by applying well-known Boson-expansion techniques. That such techniques, which are highly developed in nuclear physics, can be very useful for interacting Bose systems was first demonstrated by (CDDF-BPD) [15, 16]. We have applied this approach, for the first time, in the context of field theory. In a systematic $1/N$-expansion it has become possible to derive a $\pi\pi$-scattering equation, to lowest order in $1/N$. This scattering equation is of RPA type and fulfills all required properties such as unitarity, Ward identities and standard RPA sum rules. The formalism is manifestly non-covariant. However the full covariant theory collapses to a two times-theory in this order of the $1/N$ expansion. Therefore no breaking of covariance occurs in the final results. Furthermore special care had to be taken in dealing with Goldstone modes, which is crucial for $\pi\pi$-scattering in the chiral limit. Due to the symmetry-conserving character of the approximations these aspects are treated correctly in the present approach. The lowest-order equation disregards u- and t-channel exchanges in the $I = J = 0$ channel and no physically reasonable parameters (cut-off and sigma mass) could be found to reproduce the empirical $\pi\pi$ phase shifts. Contributions from u- and t-channel exchanges , possibly curing this problem, enter only in next-to-leading order and will be studied in a forthcoming publication.

One can envision several other applications of interest. A direct extension of the present study would be the inclusion of fermionic degrees of freedom to treat the $\pi - N$ system on a new level, including non-perturbative $\pi\pi$ rescattering effects systematically. Another area concerns the thermodynamics of the linear sigma-model. In ref. [1] the finite-temperature version of the HFB-QRPA in $O(4)$ has been studied, in an attempt to describe the chiral phase transition. This transition was found to be of first-order. To leading order in the $1/N$-expansion, on the other hand, it turns out to be of second-order, as will discussed in future work. Similar to the $\pi\pi$-scattering case it will be important to see what the next-to-leading-order corrections are.

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7 Appendix 1

In this appendix we want to show how one can handle safely the square roots originating from the Holstein-Primakoff mapping. Obviously one rather needs the expanded form of these functions in terms of the operators A and A^+. Thus one could think of performing a formal Taylor expansion. However since these operators are non commuting one therefore must be careful when proceeding with the expansion. The major difficulty comes from the operator differentiation needed for the evaluation of each order in the Taylor expansion. Several methods have been designed for this purpose. Among these methods there exists a very simple and powerful one based on the parameter differentiation technique (see for instance [27]). Here we will present a slightly different derivation.
The starting point is the parameter differentiation of the exponential. Given a function z of the complex variable λ then the following identity holds:

$$\frac{\partial e^{z(\lambda)}}{\partial \lambda} = \int_0^1 d\alpha \; e^{(1-\alpha) z} \, \frac{\partial z(\lambda)}{\partial \lambda} \, e^{\alpha z} \tag{96}$$

The identity above was derived by Snider in [28] and can be recovered, following the same author, by first recalling the relation:

$$\frac{\partial z^n}{\partial \lambda} = \sum_{m=0}^{n-1} z^m \, \frac{\partial z}{\partial \lambda} \, z^{n-m-1} \tag{97}$$

which can be shown to hold by a simple induction. Using this result the left hand side of eq.(96) reads:

$$\frac{\partial e^{z(\lambda)}}{\partial \lambda} = \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \frac{1}{n!} z^m \, \frac{\partial z}{\partial \lambda} \, z^{n-m-1} \tag{98}$$

Making now the substitution: $p = n - m - 1$, one observes that

$$\frac{m!\,p!}{(p+m+1)!} = B(p+1\,,\, m+1) = \int_0^1 (1-\alpha)^m \, \alpha^p \, d\alpha \tag{99}$$

where $B(x, y)$ is nothing else but the beta function. Inserting now the above in eq.(98) one gets:

$$\begin{aligned} \frac{\partial e^{z(\lambda)}}{\partial \lambda} &= \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{m!p!}{(p+m+1)!} \frac{z^m}{m!} \, \frac{\partial z}{\partial \lambda} \, \frac{z^p}{p!} \\ &= \int_0^1 d\alpha \sum_{n,p=0}^{\infty} (1-\alpha)^m \; \alpha^p \frac{z^m}{m!} \, \frac{\partial z}{\partial \lambda} \, \frac{z^p}{p!} \end{aligned} \tag{100}$$

Now one can easily group the terms together and check that this is effectively the desired result of eq.(96).

Back to our problem of the square root expansion, we wish to make use of the identity derived so far. First to generate an exponential out of the square root one simply recalls the inverse Laplace transform:

$$\sqrt{t} = \frac{1}{4\sqrt{\pi}i}\int_{c-i\infty}^{c+i\infty} e^{ts}s^{-\frac{3}{2}}ds \tag{101}$$

Expanding the Holstein-Primakoff square root amounts then to expanding the exponential of the same argument times an arbitrary parameter s. For the purpose of the parameter differentiation we define the argument of the square root (or of the exponential) as:

$$t \;=\; t(\lambda) \;=\; N \;+\; \hat{d}^2 \;+\; \lambda\left(\hat{d}\tilde{A} + \tilde{A}^{+}\hat{d} + \tilde{A}^{+}\tilde{A}\right) \tag{102}$$

For $\lambda = 1$ one recovers the argument of the HP square root. We recall that the matrix $\hat{d}$ is diagonal,and that it does not commute with the operators $\tilde{A}$ and $\tilde{A}^{+}$. The fact that it scales like $\sqrt{N}$ is crucial because it helps to organize the expansion in the terms of powers of $\sqrt{N}$ which is the real parameter of the expansion (see text).
Gathering everything together one can write finally the HP square root into the following Taylor expansion:

$$\left(\sqrt{t(\lambda)}\right)_{q,p} = \frac{1}{4\sqrt{\pi}i}\int_{c-i\infty}^{c+i\infty} s^{-\frac{3}{2}}ds\left(\left[e^{st(\lambda)}\right]_{\lambda=0} \;+\; \lambda\left[\frac{\partial e^{st(\lambda)}}{\partial\lambda}\right]_{\lambda=0} \;+\; \frac{\lambda^2}{2}\left[\frac{\partial^2 e^{st(\lambda)}}{\partial\lambda^2}\right]_{\lambda=0} \;+\;\ldots\right)_{q,p} \tag{103}$$

As an application of the above considerations one can, through a trivial intermediate algebra, show that the lowest orders (used in this paper) for the HP square root read:

$$\begin{aligned}
\left(A^{+}\sqrt{N\,+\,A^{+}A}\right)_{q,p} \;&=\; N\left[d_q\sqrt{1+d_q^2}\,\delta_{q,p}\right] + N^{\frac{1}{2}}\left[\tilde{A}^{+}_{q,p}\sqrt{1+d_p^2} \;+\; d_q\frac{\sqrt{1+d_q^2}-\sqrt{1+d_p^2}}{d_q^2-d_p^2}\left(d\tilde{A}+\tilde{A}^{+}d\right)_{q,p}\right] \\
&+\;\left[\sum_m \frac{\sqrt{1+d_m^2}-\sqrt{1+d_p^2}}{d_m^2-d_p^2}\left(\delta_{q,m}\,d_q\left(\tilde{A}^{+}\tilde{A}\right)_{q,p} \;+\; \tilde{A}^{+}_{q,m}\left(d\tilde{A}+\tilde{A}^{+}d\right)_{m,p}\right)\right. \\
&+\;\left.\sum_m \mathcal{F}(q,m,p)\frac{d_q}{2}\left(d\tilde{A}+\tilde{A}^{+}d\right)_{q,m}\left(d\tilde{A}+\tilde{A}^{+}d\right)_{m,p}\right] \;+\; \theta(N^{-\frac{1}{2}})\ldots
\end{aligned} \tag{104}$$

with the following definition of the function $\mathcal{F}(q,m,p)$:

$$\mathcal{F}(q,m,p) = \frac{\sqrt{1+d_q^2}+\sqrt{1+d_p^2}-2\sqrt{1+d_m^2}}{(d_q^2-d_m^2)(d_m^2-d_p^2)} + \frac{\sqrt{1+d_q^2}-\sqrt{1+d_p^2}}{d_q^2-d_p^2}\left(\frac{1}{(d_q^2-d_m^2)}+\frac{1}{(d_p^2-d_m^2)}\right) \tag{105}$$

One can further simplify these expressions by using the symmetry property of the bosons $\tilde{A}, \tilde{A}^+$ under the interchange of the indices q and p. The same expansion can be performed for the hermitian conjugate of the right hand side of eq.(104). These two expansions are the building blocks of the Hamiltonian in eq.(28).

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